A MODEL FOR AGGREGATE INVENTORY CONTROL OF RETAIL DRUGS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Richard Fussell Ward

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Industrial Engineering

Georgia Institute of Technology

August, 1964

A MODEL FOR AGGREGATE

INVENTORY CONTROL OF RETAIL DRUGS

Approved:

Chairman Dr. John H. MacKay

Dr. Robert N. Lehrer

Dr. George E. Maddox

Dr. Harrison M. Wadsworth

Date approved by Chairman: 8-24-64

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

ACKNOWLEDGMENTS

The inspiration for this study resulted from an awareness of the need for practical inventory controls which are applicable to an aggregate of items. Interest in this area was imparted to the author by the faculty of the School of Industrial Engineering.

Special credit is due Dr. John H. MacKay, the thesis committee chairman, for his challenging interest and valuable guidance throughout the study. His energetic efforts during the most difficult of times are sincerely appreciated by the author.

Special credit is also due Dr. George E. Maddox, Dr. Robert N. Lehrer, Dr. Harrison M. Wadsworth and Associate Professor E. C. Franklin for their suggestions on the manuscript.

Finally, I acknowledge and shall always be grateful for the cooperation and encouragement given by my wife, Jere and daughter, Delaree during the time of my academic work.

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

The objective of this study was to develop purchasing policies on an aggregate basis for drugs in a retail store. The purchasing policies that were developed specify the economic order quantity and the reorder point stock level.

The Jacobs Drug Company of Atlanta, Georgia, cooperated in providing data from its current records which were required to authenticate the study. Patent medicines served as the class of drug items upon which this study was based. Total inventory cost was selected as the measure of effectiveness for constructing the model.

The model constructed in this study provides the managers of stores in the retail drug industry with measures upon which to base inventory policies. Specifically, the model determines the following relations:

1. The average dollar investment in cycle inventory for various levels of ordering frequency.
2. The dollar investment in safety inventory for various levels of ordering frequency.
3. The dollar investment in safety inventory for various levels of average probability of a stock-out for all items.

The determination of cycle inventory requirements was based on an analysis of the total cycle inventory cost function. Using this function,

a series of cycle inventory policies was derived for various levels of average total dollar investment in cycle inventory and the total number of replenishment orders each period. A stochastic analysis of the cost to place a replenishment order and the cost to maintain an item in inventory was used to estimate a range of policies within which is contained the policy that minimizes the total cost function.

An analysis of the combined distribution of demand during lead-time was used to determine the safety inventory requirements for various levels of average probability of a stock-out for all items. With the information about the cycle and the safety inventory requirement, a series of purchasing policies was developed. Based upon the information presented through the series of purchasing policies the store manager was able to select the purchasing policy which minimized total inventory costs and fulfilled his objectives with regard to inventory investment, customer service and ordering frequency.

Results of the study were as follows:

1. Sales activity of patent medicines is such that an aggregate of items can be grouped together for inventory control.
2. The inventory model can provide an accurate guide to evaluate the various inventory policies.
3. A statistical analysis of variable cost factors can be used to estimate a confidence interval of inventory policies

within which is contained the policy that minimizes total inventory cost.

4. To improve the accuracy of the model all relevant factors should be considered where possible.

CHAPTER I

INTRODUCTION

The objective of this study is to develop purchasing policies on an aggregate basis for drugs in a retail store. The purchasing policies will specify the economic order quantity and reorder point stock level.

Patent medicines were selected as the class of drug items upon which to base this study. Data were collected at a Jacobs Drug Store in Atlanta and total inventory cost was selected as the measure of effectiveness for constructing the model.

General Need for Inventory Control

Inventory analysis techniques provide a means for measuring the variable factors associated with maintaining an inventory. These variable factors include:

1. Total inventory carrying cost.
2. Total ordering cost.
3. Total cost of a given service level.
4. Dollar investment in inventory.
5. Frequency of placing a replenishment order.

Only to the extent that these variables are subject to reliable quantitative measurement is it possible for effective control procedures to be established. Once this has been done, it is possible to incorporate the inventory function into the overall management plan.

Background

The trend in recent years has been toward diversification and increased variety in industrial production. Consequently, manufacturers, wholesalers, and retailers have developed large inventories which, in turn, have created a demand for the development of reliable methods of control. This need for inventory control is further emphasized by the recent increase in the percentage of the total assets of manufacturing and retail trade industries which are contained in inventories. Since 1953 this percentage has ranged from twenty to fifty per cent among these industries.[1]

The demand for a great variety of products has created a critical problem in certain retail trade industries. Typically, drug stores, super markets and department stores have experienced such increased demand. These stores are noteworthy because a major portion of their inventories consists of a large number of low value items. The value characteristics of these inventories are not adaptable to the item-by-item analysis which is most frequently used in inventory control. Since the logical approach

[1] Crowell, Thomas J., Economic Almanac 1958-1960, New York.

to an inventory analysis is to consider the high value items first, these low value items are often controlled by an arbitrary policy which is not based on objective analysis.

Consulting with several druggists and observing their present operating practices, revealed a need for improved inventory control methods in the retail drug industry. The primary need relates to the establishment of objective purchasing policies. Customary management practice in the retail drug industry requires the store owner or manager, who is frequently a druggist, to establish the operating policies of the store. Under this type of management, the stores observed were found to be utilizing purchasing policies which were not susceptible to evaluation based on objective measurements.

In privately-owned stores, the order quantity discounts are the prime consideration in determining how much to buy. This price-break motivation often causes the druggist to order large quantities of items that are not needed. The manager-druggist of the chain-owned store does not have the price-break motivation. In the latter case all drug items are ordered directly from the company' s central warehouse. Order size and frequency of ordering only depend on the judgment of the store manager.

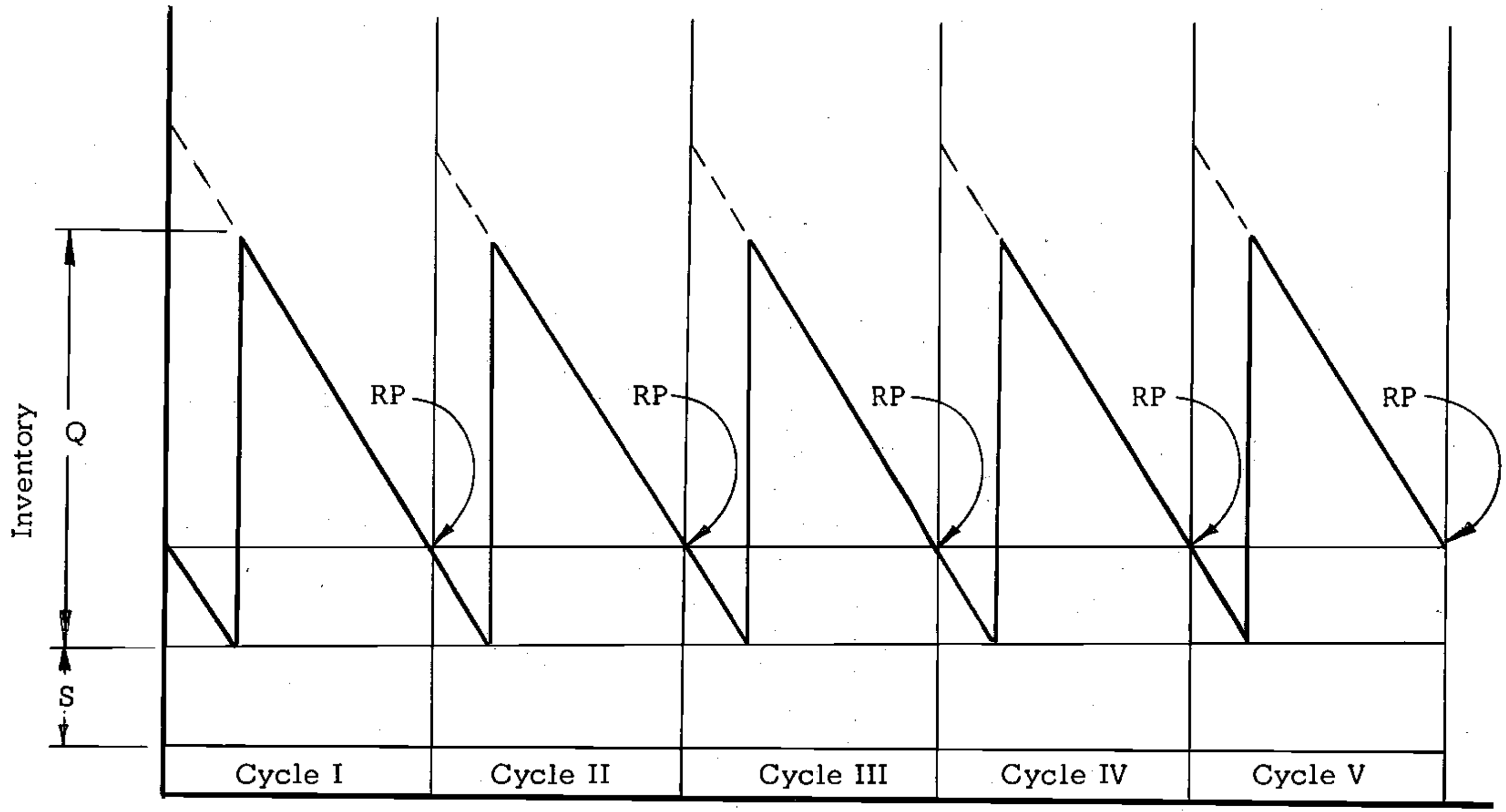
General Framework for Inventory Control

In order to provide the retail drup industry with measures upon which to found inventory policy, a model is needed which, based on economic lot sizes and protective stock levels derived from stochastic processes, determines the following relations:

1. The average dollar investment in inventory corresponding to various levels of replenishment order size.
2. The average dollar investment in inventory corresponding to various levels of ordering frequency.
3. The dollar investment in inventory corresponding to various levels of customer service.

The fixed quantity inventory policy was selected as a general framework within which to develop purchasing policies for the selected drug items. The fixed quantity policy, which is illustrated in Figure 1, requires that a replenishment order be placed for a fixed quantity (Q) whenever the inventory on hand and on order is depleted to a predetermined level. The assumptions of this policy are:

1. Stock level surveillance is continuous.
2. A replenishment order can be placed at any time.
3. Demand and lead-time are constant and known.
4. Receipt of stock into inventory is instantaneous.
5. Procurement cost is constant for each item.
6. The cost to maintain an item in inventory for a fixed period of time is constant for each item.



Q - Fixed Order Quantity
S - Safety Inventory
RP- Reorder Point

—— Inventory Level
- - - Inventory on Hand and on Order

Figure 1. Graphical Illustration of the Fixed Quantity Inventory Policy.

7. The inventory is depleted in small increments and no back orders are allowed.

8. The items as grouped for an aggregate analysis have similar inventory carrying and procurement costs.

In the remainder of this study, a cycle will refer to the time interval between placement of subsequent orders for an individual item, and one half the value of the quantity ordered (i.e., fixed order quantity) each cycle will be referred to as the working inventory investment. A period is the time interval for which a purchasing policy is developed; for this study the period is one half year. Usually this period will extend over several cycles. The term "stock-out" will refer to the case where demand cannot be satisfied because the inventory level is zero. The term "lead-time" will refer to the time interval beginning when an order is entered in the weekly order book and ending when the order is received at the store.

The reorder point is a predetermined inventory level established to provide for the expected demand during lead-time plus an additional amount for unexpected variation in the demand during lead-time. The amount of inventory provided to protect against a stock-out resulting from unexpected variation in the demand during lead-time will be referred to as "safety stock."

Scope of Study

The present study will be concerned with the determination of purchasing policies which are applicable to patent medicines in a retail drug store. The fixed quantity inventory policy will be used as a general framework within which the purchasing policies will be formulated. Total relevant inventory costs which are the costs of maintaining and replenishing inventory, and the cost of under stocking inventory will serve as the measure of effectiveness for the purpose of constructing a model.

The total inventory cost will be considered simultaneously with the total annual demand for an aggregate of items in the development of a continuous series of working inventory purchasing policies. A stochastic treatment of the inventory costs will be used to determine an estimate of the range of purchasing policies within which lies the policy that minimizes the total inventory cost relation.

The safety stock requirements of this model are based on a statistical evaluation of the combined actual demand and the lead-time distribution functions. A continuous series of safety stock requirements for various levels of probability of a stock-out will be developed.

The working inventory and safety inventory will be combined for an analysis of the total inventory investment required for various combinations of order frequency, order quantities and risk of stock-out.

General Assumptions

The following assumptions will be made through this study:

1. Stock level surveillance is continuous.
2. A replenishment order is placed when it is entered in the weekly order book.
3. A replenishment order can be placed at any time.
4. Other assumptions will be listed as they apply to specific relations.

Limitations

The methods of analysis employed in this study are applicable to inventories composed of a large number of items with similar cost factors. Items with cost factors varying substantially from the mean should be considered on an equivalent order basis or for individual item control.

CHAPTER II

LITERATURE SURVEY

Prior to the early nineteen-fifties, developments in inventory control were slow. Beginning with the efforts of Federick W. Taylor to improve industrial practices, inventory control became recognized as an area for substantial cost reduction. Raymond (11)[1] made the most significant contribution during the period of early development. He formulated mathematical relations which included all of the factors which might conceivably affect the economic lot size. Most of the work of this period assumed the elements of inventory control to be deterministic in nature. Since this early period, the economic lot size formula has been modified to consider the dynamic nature of the elements of the inventory problem under stochastic conditions where the uncertainty of relevant factors are analyzed on a probabilistic basis.

Recent texts have devoted considerable attention to the buffer or safety stock required for various levels of protection against stock-outs. Vazsonyi (13) presents several stochastic models for determining the safety stock required when demand is of a known distribution. He assumes

[1] Numbers in the parentheses refer to references listed in the bibliography.

replenishment lead-time to be deterministic and inventory costs to be constant and known.

A major portion of the work by Arrow, Harris and Marschak (1) is connected with the (S, s) inventory policy. The authors develop a series of models for both deterministic and stochastic situations. They assume deterministic replenishment lead-time and complete knowledge of inventory costs.

Churchman, Ackoff, and Arnoff (3) present various inventory models involving deterministic and stochastic processes. In the mathematical expression for the economic order quantity, the authors provide a means for considering the effects of variable unit cost, obsolescence and storage limitations. The assumptions of their models are deterministic replenishment lead-time and knowledge of inventory costs.

Fetter and Dalleck (8) have compiled a collection of inventory models covering some practical phases of inventory control. They modify the classical economic order quantity formula for various constrained conditions. In their treatments of safety stock, they combine the demand and replenishment lead-time distributions into demand during lead-time. This represents a situation which frequently occurs in inventory operations. The basic assumption of their models is knowledge of inventory costs.

Sasieni, Yaspan, and Friedman (12) developed inventory models for conditions of variable demand and deterministic lead time. The authors developed their models on the basis of assumed known cost of a stock-out, cost to maintain an item in inventory, and cost to place a replenishment order.

Arrow, Karlin and Scarf (2) summarize the inventory models involving elements of a stochastic nature. The author's treatment of various demand and replenishment lead-time density functions are extended to cover dynamic inventory situations. The primary assumption of these models is knowledge of inventory costs.

A shortcoming common to all of the above listed inventory control studies is that they deal with decisions to be made for single items. In controlling inventory on an item-by-item basis each item is analyzed independently. Control on an aggregate basis takes into consideration the interactions of the items. These interactions take the form of competition among items for a fixed amount of funds available for obtaining and holding stock, subject to the physical constraints of limited purchasing department capacity and storage facilities. Feeney (7) proposes a method for presenting a continuous series of inventory plans. Each plan is subjected to given management objectives. These objectives may be in the form of aggregate restrictions such as the total number of orders, the average inventory investment for each period and the maximum allowable number of stock-outs.

Welch (14) developed a method of control similar in mathematical analysis to that of Feeney, which is based on demand being fixed and known. He considers items that can be classified into broad categories with similar values for the cost parameters of their economic order quantity formulas. Noteworthy is his analysis of an aggregate inventory of substantially different items based on equivalent values for the cost parameters of the economic order quantity formula.

Davis (4) developed decision rules which are applicable to large inventories. His rules are designed to consider demand and replenishment lead-time uncertainty, cost of a stock-out, ordering and handling costs, obsolescence and physical holding costs and relative essentiality of the product. However, the magnitude of the calculation required to determine the value of the parameters for these factors on an individual item makes his system impractical for inventories of a large number of low value items.

All of the references cited, excluding Welch and Feeney, base the mathematics of their models on the assumption that cost parameters are known. Primarily, these cost parameters are the cost to carry an item in inventory (carrying cost), the cost to place a replenishment order (ordering cost) and the cost of a stock-out. Since such cost expressions are difficult to determine, inventory models based on such expressions are difficult to apply to inventories composed of a large number of low value items. Incorporated within any model using these cost expressions is the assumption that the objective of management is minimization of these costs. Such an assumption is not always valid.

The literature search revealed that inventory control on an aggregate basis has received limited consideration. The present study will develop a method of selecting purchasing policies for an inventory composed of a large number of low value items.

Since the selection of a purchasing policy is dependent upon knowledge of inventory costs, physical limitations and management objectives, no single policy will be recommended. A continuous series of purchasing policies will be developed which are independent of knowledge of inventory cost parameters. A technique for estimating inventory costs on a gross basis will also be developed for the purpose of establishing a range of purchasing policies within which is contained the policy that minimizes the total cost relation.

CHAPTER III

EXPERIMENTAL PROCEDURE

Introduction

In approaching the problem of inventory control in the drug industry it was necessary to secure the cooperation of the management of a store maintaining an inventory with the desired characteristics. It was also considered desirable to use the current records of an active business operation in order to test the practical aspects of the proposed plan of control. One of the several stores owned by the Jacobs Pharmacy Company consented to provide the data required to authenticate the study.

The characteristics of the class of inventory included in the study are low value items, (i.e., less than three dollars) and large numbers of different items (i.e., more than fifty). The two above factors plus the accessibility of essential information were the criteria used in selecting the class of items for study. After evaluating the characteristics of the 15 classes of inventory items sold in the drug store, a class of patent medicines was selected.

Presented in the remainder of this chapter are the general concepts of inventory control as found applicable to the situation under study.

In addition, all mathematical formulae used in the development of the inventory concepts are derived, referenced or justified herein. Actual data is not included as a part of this chapter, presentation and analysis of data are included in Chapter IV, "Analysis of Data."

Model Formulation

In order to determine the degree of control required by each item in the inventory under study, a distribution by value analysis[1] was performed (see Figure 2).[2] Based on a relative measure of average period dollar demand for all items, there was insufficient reason to exclude any of the items in the inventory from aggregate control analysis.

The fixed quantity model selected for application to the case under study facilitated the analysis of the total inventory investment on the basis of its components, these components being working and safety inventory.

Work Inventory Analysis

The formula derived for determining the order quantity for each item is based on an analysis of the relevant cost factors associated with obtaining and maintaining inventory. Beginning with the total relevant inventory cost function, the following relation is derived:[3]

[1] Welch, op. cit., p. 19.

[2] The figures referenced in this chapter are developed and presented in Chapter IV, "Analysis of Data."

[3] Appendix A.

$$\sum_{i=1}^{t} (A_i)(N_i) = \frac{\left(\sum_{i=1}^{t} \sqrt{D_i C_i}\right)^2}{2} \tag{1}$$

where A_i = average dollar investment in working inventory for the i^{th} item

N_i = number of replenishment orders placed each period for the i^{th} item

D_i = demand each period for the i^{th} item in units

i = item number

t = total number of items in the inventory

C_i = unit cost of the i^{th} item

The assumptions required for the derivation of this relation are as follows:

1. The demand for each period is constant and known.
2. The procurement cost is constant for each item.
3. The cost to maintain an item in inventory for a fixed period of time is constant for each item.
4. The inventory is depleted in small increments and no back orders are allowed.
5. The items as grouped for an aggregate analysis have similar inventory carrying and procurement cost.

The assumption of constant and known demand which is required for the derivation of equation 1 is not satisfied for the case under study as in many other actual situations. However, based on past activity and the premise that the past activity gives knowledge of future activity, the assumption is justified. The remaining assumptions are justified without further qualification.

Relation 1 takes the form of a continuous surface when graphed on coordinates of average dollar investment in working inventory versus total number of replenishment orders each period. Thus, knowing only the period demand for each item and the unit cost of each item, it is possible to describe the relation between average dollar investment in inventory and the total number of replenishment orders necessary each period to minimize the total relevant inventory cost relation.

Equation (1) was evaluated for the items under study and the results presented graphically (see Figure 3) on coordinates of average dollar investment in working inventory versus the total number of orders each period.

The working inventory analysis fulfilled all of the requirements for this inventory component and, thus, was considered adequate for the case under study.

<u>Safety Inventory</u>

The safety inventory requirements for an aggregate of items is based on an analysis of the demand and lead-time distribution.

<u>Lead-time and Demand Distribution Analysis</u>. A chi-square test was used to compare the actual demand distribution with the theoretical normal distribution having the same mean as the observed distributions. Using a forty per cent random sample of the items under study, the hypothesis that the observed distributions were from a normal distributed population was accepted at the five per cent significance level for twenty per cent of the items compared. The demand distributions of the same items were then compared with the theoretical Poisson distribution having the same mean as the observed distributions. The hypothesis that the observed distributions were from Poisson distributed populations was accepted at the five per cent significance level for 82 per cent of the items compared.

An investigation of the items which were rejected in the Poisson comparison revealed that most of these items were either of such low unit value that the store manager ordered large quantities at infrequent intervals and the data used would not reveal actual demand or the demand was inadequate to provide a valid chi-square test. Based on the results of the previous investigation, it was assumed that the demand distributions for the patent medicines considered in this study were from Poisson distributed populations.

There were insufficient data to determine the characteristic of the replenishment lead-time. However, it was known that lead-time can vary from one to seven days, because of service policies outside the control of the store management. The theoretical Poisson distribution was arbitrarily selected to approximate the lead-time variable.

Using the theoretical Poisson distribution to describe the demand and an assumed theoretical Poisson distribution to describe the lead-time variable, the combined distribution of demand during lead-time function[4] was evaluated for a mean lead-time held constant at three days. The mean lead-time was arbitrarily set at three days. Although the true mean lead-time may not be three days, this does not alter the validity of the operational characteristics of this model. If the mean lead-time was less than the assumed three days, safety inventory investment could be reduced and the desired customer service level maintained.

Results of the evaluation of the combined demand and lead-time function were presented graphically for various levels of average usage during lead-time on coordinates of reorder point inventory level versus average probability of a stock-out.

Safety Inventory Investment Distribution. The safety inventory investment for an aggregate of items should be distributed among the individual items in a manner that will yield the maximum return on investment in terms of customer service. This may be interpreted as a management objective of reducing both safety inventory investment and total number of

---

[4] Ekey, Talbird and Newberry, *op. cit.*, p. 33.

stock-outs. Inorder to achieve this objective, the distribution of safety inventory for an aggregate of items is such that fewer stock-outs will occur for inexpensive items than for expensive items. Such a distribution seeks to prevent excessively large amounts of money from being invested in safety inventory for high priced items and thus, each marginal dollar invested in safety inventory will tend to contribute equally to the reduction of shortages.

The ratio of the individual unit cost to the average unit cost was investigated as a basis for distributing the investment in safety inventory such that the desired relationship is achieved between the various cost items. In order to dampen the effect of the extreme individual unit cost values, the ratio was raised to the 0.50 power. Other powers were investigated for the range of unit cost values of patent drugs, however, the 0.50 power appeared to give satisfactory emphasis to the low cost items. Based on this logic the following relation was determined:

Where

$P(S_i)$ is the probability of one or more stock-outs of the i^{th} item during a period and

$\overline{P(S)}$ is the average probability of one or more stock-outs for all items during a period[5]

[5]This represents an average or uniform probability level which is intended to be specified by management and representative of the service level which is acceptable for the class of items under study. An example of this would be management specifying an acceptable probability of one or more stock-outs during a period equal to 0.10 for all items. Interpreted, this would mean that for item No. 1 the $\overline{P(S)} = 0.10$, for item No. 2 the $\overline{P(S)} = 0.10$, etc.

C_i is the unit cost of the i^{th} item

$\overline{C}$ is the average unit cost for all items.

$$P(S_i) = \overline{P(S)} \sqrt{C_i/\overline{C}} \tag{2}$$

It is emphasized that Equation (2) is an arbitrarily selected relation and the purpose of multiplying the average probability by the square root of the ratio of the unit cost of the i^{th} item to the average unit cost is to adjust the average probability in order to achieve the desired distribution of aggregate safety inventory among the individual items.

Probability of a Stock-Out. The only time at which a stock-out can occur is during the replenishment lead-time.[6] Therefore, the probability of a stock-out during a period must be converted to the probability of a stock-out during lead-time. This conversion is also required in order to apply the results of the analysis to the demand and lead-time distribution function (presented graphically in Figure 6).

If $P(S_i)$ is the probability of one or more stock-outs for the i^{th} item during a period; if $P(S_i)^*$ is the probability of one or more stock-outs for the i^{th} item during any one lead-time; and if n_i is the number of replenishment orders placed during a given period for the i^{th} item, then;[7]

---

[6]Fetter and Dalleck, op. cit. p. 14.

[7]Appendix B.

$$P(S_i)^* = 1 - [1 - P(S_i)]^{\frac{1}{n_i}} \tag{3}$$

The assumptions of this equation are:

1. The demand for individual items is independent during each reorder cycle.
2. The demand for each item during each reorder cycle is independent of the demand during every other reorder cycle.

Equation 3 was evaluated for the values of $P(S_i)$, obtained from Equation (2), at various levels of the total number of replenishment orders each period. The resulting values were referred to the demand and lead-time distribution analysis (presented in Figure 6) and the reorder point and safety inventory requirements determined.

Safety Inventory Investment

The reorder point inventory level for each item was determined by referring the values of $P(S_i)^*$, described in the previous paragraph, to the demand and lead-time distribution analysis (presented in Figure 6). The safety inventory requirements for the individual items were determined by subtracting the average usage during lead-time from the reorder point inventory level. The resulting safety inventory requirements were aggregated over all items.

The results of the safety inventory analysis were presented graphically on coordinates of average probability of a stock-out versus safety inventory investment.

Combined Inventory Investment

With the analysis of safety and working inventory investment complete, the results were combined for an analysis of the total inventory investment. The results of the analysis were presented graphically for various levels of replenishment ordering frequency on coordinates of average probability of a stock-out versus total inventory investment.

The graphical results of the combined analysis were presented to the manager of the store in which the study was conducted. The manager was invited to select the total inventory investment, ordering frequency, and acceptable customer service level which he considered best fulfilled his management objectives with regard to the class of drug items under study. Upon selecting a value for each of the factors, the total estimated inventory carrying and ordering costs associated with the plan selected were determined along with the reorder point and reorder quantity for the individual items.

Data Collection

The weekly ordering records of the drug store were used as the source of a major part of the data required for the analysis. These records were considered by the store manager to provide a good indication of the demand for all 15 classifications of items sold in the store. After evaluating all 15 classifications of items, a class of patent medicines was selected for study. This class of medicine, contained 550 different items. In order to reduce the required calculations for the purpose of this study, a 25 per cent random sample was taken from the total 550 items. There were 138 items selected.

The data needed to determine the working inventory requirements are period demand and unit cost for each item. The demand information was obtained from the weekly ordering book for a period of 26 weeks. Data for a longer period of time were sought in order to provide a better indication of actual demand. However, such data were not available in the store's records.

Exact determination of the cost factors (i.e., procurement cost and cost to maintain an item in inventory) was not attempted in this study. A stochastic analysis of these factors was conducted and the extreme values used to estimate a range of working inventory ordering policies within which lies the policy that minimizes the total cost relation. The data necessary for this phase of the study were obtained from estimates of the cost factors developed by the drug store manager.

The distribution of weekly demand as obtained from the weekly order books was used as a basis for estimating the demand distribution for the patent medicines. The weekly ordering records provided a satisfactory indication of the demand distribution because under existing management practice an effort was made to maintain a certain level of inventory for each item. In addition, orders were placed at frequent intervals, thus increasing the correspondence between the ordering activity and demand. Since actual observation of demand was considered impractical, and although the ordering records were not a perfect indication, there were no data available to provide a better indication of demand.

The existing ordering procedures were investigated to determine the characteristics of this variable. Assuming that an order is placed when the store manager enters it in the weekly order book, replenishment lead-time was found to be a discrete variable taking on integral values of from zero to seven days.

CHAPTER IV

ANALYSIS OF DATA

Selection of Drug Items for Study

The class of patent medicines selected for this study contained 550 different items. In order to reduce the number of calculations required in the data analysis, a random sample of 25 per cent of the items was selected. The mechanics of the sampling technique involved numbering each of the 550 items in sequential order. Matching the last three digits of a five digit random number table with the number assigned to each item, 138 items were selected. The items selected in the sample are presented in Table 2 of Appendix E.

Distribution by Value Analysis

The distribution of dollar demand among the selected items was analyzed for the purpose of stratifying the sample into the items requiring various degrees of control. The analysis required data on the total period demand in units and the unit cost of each item, which was obtained from the weekly order books and presented in Appendix E in Table 2. The period dollar demand for each item was calculated and arranged in descending order of magnitude. After arranging the items, the cumulative per cent of total period dollar demand and cumulative per cent of total items was calculated and presented in Appendix E in Table 2. The results are also presented graphically in Figure 2 on coordinates of cumulative per cent of total period dollar demand (ordinate) versus cumulative per cent of total number of items (abscissa).

In evaluating the distribution by value curve of Figure 2, the top twenty per cent of the items were considered for individual inventory control. However, based on the criteria of low unit value and relatively low cumulative per cent of total dollar demand[1] for the top twenty per cent group, there was insufficient reason to exclude this group of items from aggregate control. Since the high twenty per cent group was accepted, all of the 138 items were included for aggregate inventory control.

Cycle Inventory Analysis

The cycle inventory investment in a fixed quantity inventory model is one half of the order quantity. The order quantity, in turn, is proportional to the square root of demand.[2] The exact proportionality for an aggregate of inventory is dependent on management's investment objectives and limitations.

Cycle Inventory Model

The order quantity formula in the fixed quantity inventory model as

---

[1] Welch, op. cit., p. 19.

[2] Appendix A, Equation A2.

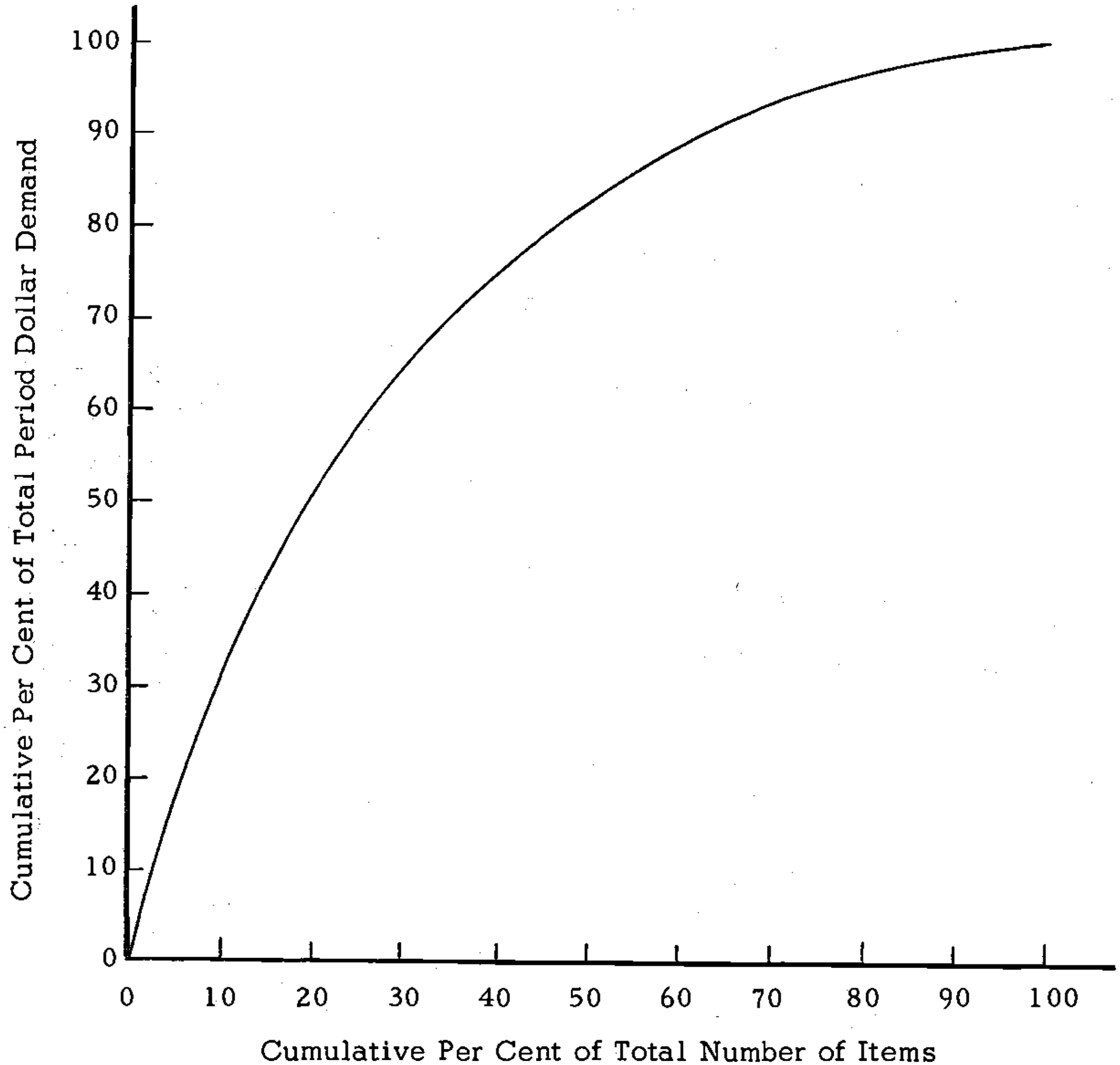


Figure 2. Distribution by Value Curve for the 138 Drug Items.

derived in Appendix A is as follows:

$$Q_i = \sqrt{\frac{2D_iS}{rC_i}} \quad . \tag{4}$$

Equation (4) can be separated into a fixed and a variable part as follows:

$$Q_i = \sqrt{\frac{2S}{r}} \sqrt{\frac{D_i}{C_i}} \quad . \tag{5}$$

In a group of items having the same replenishment ordering cost (S) and per cent cost to maintain inventory (r), Equation (5) can be expressed as follows:[3]

$$Q_i = K \sum \sqrt{\frac{D_i}{C_i}} \tag{6}$$

where $K = \sqrt{\frac{2S}{r}}$. Since average inventory investment has been defined as one half of the order quantity (Q), the average inventory investment for an aggregate of items (I) can be expressed in the following terms:

$$I = \frac{\sum_{i=1}^{t} Q_i}{2} = \frac{K \sum_{i=1}^{t} \sqrt{\frac{D_i}{C_i}}}{2} \tag{7}$$

[3]Welch, op. cit, p. 32.

where t is the total number of items in the aggregate inventory.

In Equation (7) the value of (K) dictates the average aggregate inventory investment and, thus, was selected through an analysis of the relationship between the cost parameters S and r and the aggregate cycle inventory investment.

Cycle Inventory Investment

The cycle inventory investment for an aggregate of items was analyzed with the aid of Equation (1) which was discussed in Chapter III and developed in Appendix A. This equation was evaluated as follows:

$$\sum_{i=1}^{138} (I_i)(N_i) = \frac{\left(\sum_{i=1}^{138} \sqrt{D_i C_i}\right)^2}{2} \tag{8}$$

the right side of Equation (8) is a constant when evaluated for the period demand in units and unit cost data previously discussed and presented in Appendix E in Table 2. This constant value was calculated to be as follows:

$$\frac{\left(\sum_{i=1}^{138} \sqrt{D_i C_i}\right)^2}{2} = 229{,}842 \ , \tag{9}$$

Therefore,

$$\sum_{i=1}^{138} (I_i)(N_i) = 229{,}842 \ . \tag{10}$$

Equation (10) was graphed on coordinates of average aggregate investment in cycle inventory versus total number of replenishment orders each period and presented in Figure 3. The completed graph, which is called the optimum relation between orders and inventory, represents a continuous series of cycle inventory policies which specify the total number of replenishment orders required each period to maintain a given investment in cycle inventory.

Analysis of the Cycle Inventory Optimum Policy Curve

In order to aid the store management in selecting a policy for the class of items under study which would most nearly minimize the total relevant cost equation[4] and fulfill their investment objectives, the replenishment ordering and inventory carrying cost parameters were evaluated in conjunction with the graph of Figure 3. The inventory carrying cost and replenishment ordering cost were of primary interest, since the correct value for the situation under study would determine the cycle inventory policy which minimizes the total cost equation.

A stochastic analysis was used to estimate a range of the values for the cost parameters, which in turn determined a range or confidence interval of inventory policies within which lies the policy which minimizes the total cost equation. Estimates of these cost were obtained from the store manager and are presented in Table 1.

[4] Appendix A, Equation A-1.

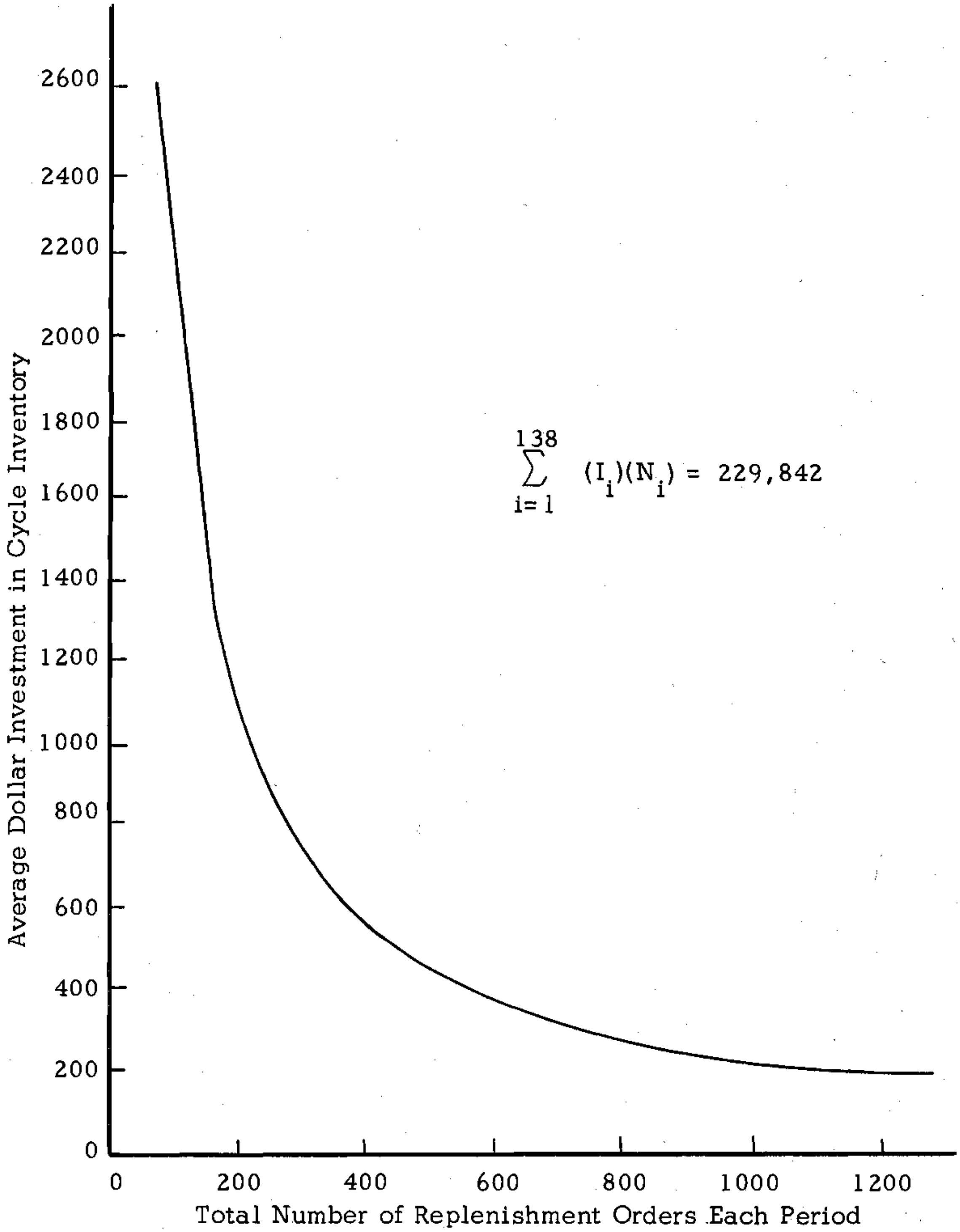


Figure 3. Graph of the Average Dollar Investment in Cycle Inventory Versus Total Number of Replenishment Orders Each Period.

Table 1. Estimates of the Replenishment Ordering and Inventory Carrying Cost

Cost	Estimated Expected Value	50% Limit Values (50% of the time cost is as high as A or as low as B)	
		A	B
Replenishment Ordering Cost - Y (per 100 replenishment orders)	$18.00	$20.00	$16.00
Cost to Carry Inventory - Z (per 100 dollars of inventory)	$14.00	$16.00	$12.00

The store manager estimated the expected values for these costs and the extreme values within which he estimates the actual values fall fifty per cent of the time. The corresponding cycle inventory policy is located on the curve of Figure 3 for a given set of cost values by determining the point of tangency of the iso-cost lines[5] and the curve of cycle inventory policies. The slope of the iso-cost lines are determined by the ratio of the cost of placing a replenishment order and the cost of carrying an item in inventory. The normal distribution was arbitrarily assigned to the distribution of these costs values. It is emphasized that the normal distribution was arbitrarily selected to approximate the distribution of these cost values and another theoretical distribution would possibly be more appropriate. However, investigation of this area is left to future study. The fifty per cent interval for the normal distribution is as follows:

[5] Welch, op. cit., p. 44.

for μ = the expected value of the distribution

σ = the standard deviation of the distribution about the expected value.

The fifty per cent interval is

$$\mu \pm 0.67\sigma . \tag{11}$$

Using the above relation, the estimated standard deviation about the expected values of the distributions of cost estimates as made by the store manager were calculated as follows:

$$\hat{\sigma}_y = 2.99$$

$$\hat{\sigma}_z = 2.99 \quad .$$

The slope of the iso-cost lines for the estimated mean or expected values of these costs is as follows:

For $\hat{\mu}_y$ = the estimated mean or expected cost to place one hundred replenishment orders.

$\hat{\mu}_z$ = the estimated mean or expected cost to maintain one hundred dollars of inventory for one period.

$$\text{Estimated expected slope} = \hat{E}\left(\frac{Y}{Z}\right) = \hat{\mu}_{\frac{y}{z}} = \frac{\hat{\mu}_y}{\hat{\mu}_z} = -1.286 \tag{12}$$

The estimated variance of the ratio of two normal distributions is:[6]

$$\hat{V}\left(\frac{Y}{Z}\right) = \frac{\hat{\mu}_y^2 \hat{\sigma}_z^2 + \hat{\mu}_z^2 \hat{\sigma}_y^2}{\hat{\mu}_z^4} = 0.121$$

The estimated standard deviation of the distribution of the ratio of the cost Y and Z is

$$\hat{\sigma}_{\frac{y}{z}} = 0.348 \ .$$

The 95 per cent confidence interval of the distribution of the slope of the iso-cost lines (i.e., the ratio of $\frac{Y}{Z}$) is as follows:

$$\hat{\mu}_{\frac{y}{z}} \pm 1.96 \, \hat{\sigma}_{\frac{y}{z}} \ .$$

Thus, the slope at the points on the purchasing policy curve that bound the ninety per cent confidence interval has a minimum value of -1.968 and a maximum value of -0.604.

The slope of the cycle inventory policy curve was determined by taking the first derivative of Equation (10) as shown below:

For I = the average aggregate cycle inventory investment

[6] Appendix C.

N = the total number of replenishment orders placed each period.

$$(I)(N) = 229{,}842$$

$$\frac{dI}{dN} = \frac{-229{,}842}{N^2} \qquad (13)$$

Solving Equation (13) for a maximum slope of -1.968 and a minimum of -0.604, the following cycle inventory policies were determined:

1. Policy at the maximum slope of the iso-cost line

$$N = 343$$

$$I = \$670.08$$

2. Policy at the minimum slope of the iso-cost line

$$N = 617$$

$$I = \$377.51$$

The points of tangency of the cycle inventory investment policy curve and the iso-cost lines for the 95 per cent interval limit values are illustrated in Figure 4.

Safety Inventory Analysis

The safety inventory investment for an individual item is dependent upon the degree of protection against a stock-out which is desired.

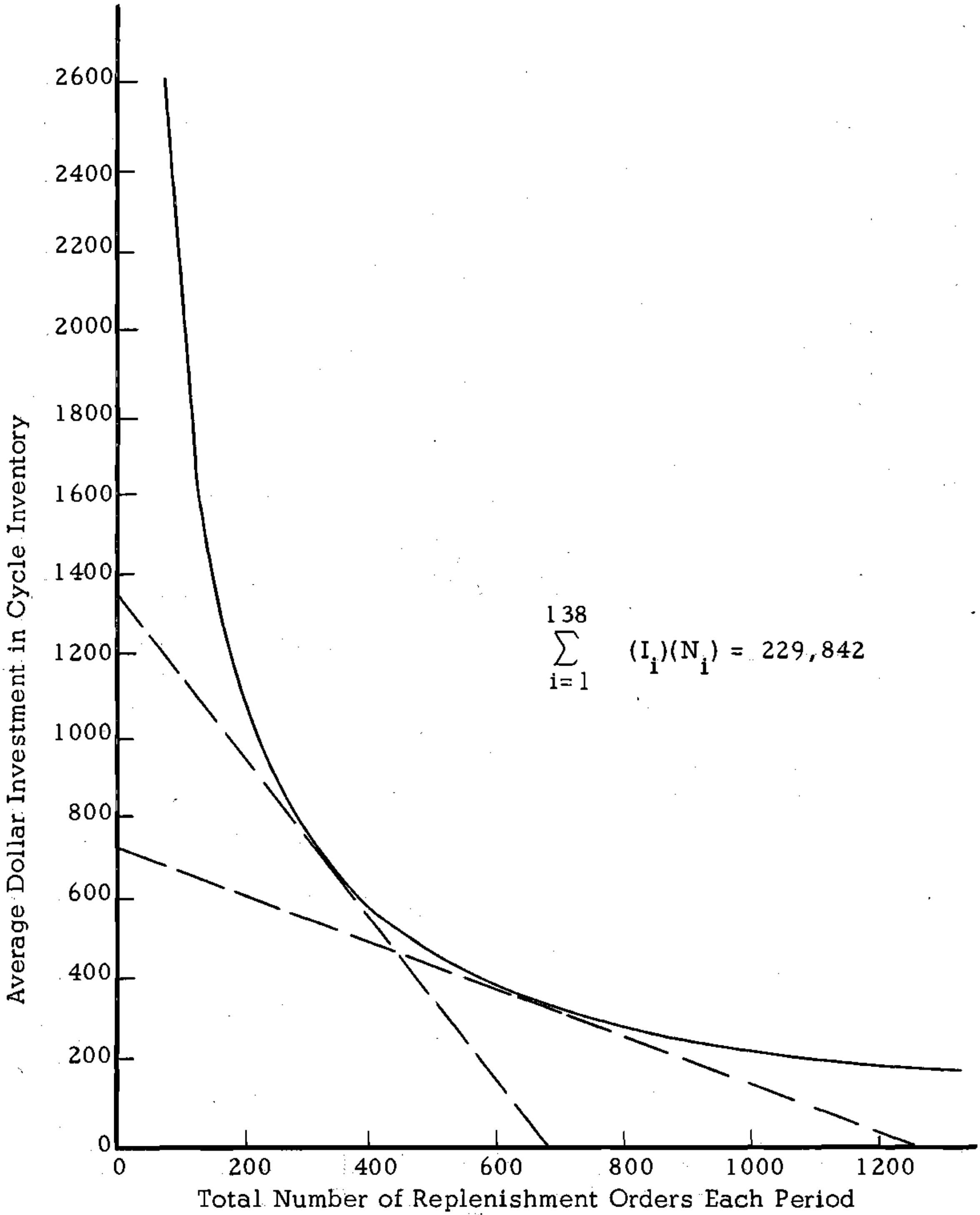


Figure 4. Presentation of the Iso-Cost Lines on the Graph of the Average Dollar Investment in Cycle Inventory Versus Total Number of Replenishment Orders Each Period.

The degree of protection is in turn influenced by the cost of protection, the potential loss from a stock-out and management's objectives with regard to customer service and associated factors. The degree of protection provided by a given investment level is a function of the characteristics of the demand and lead-time. In the case under study, the demand and lead-time, which are the principal parameters of the safety inventory model, were found to be variable.

Demand Distribution Analysis

In order to determine the characteristics of the demand distribution for the class of drug items under study, a forty per cent random sample of the 138 items was tested for goodness of fit to the theoretical normal and Poisson distributions. The Chi-square test was used for the analysis and an example is presented in Appendix D. A summary of the Chi-square test, which is presented in Table 3 of Appendix E, revealed the hypothesis that the demand distribution for the items tested were from a normally distributed population was accepted at the five per cent level of significance for only twenty per cent of the items. A test of the same hypothesis for the Poisson distribution was positive for 82 per cent of the items at the five per cent level of significance.

The items rejected in the Chi-square test for Poisson distribution were investigated to determine if there was any definable cause for rejection. In the items rejected in the test, 46 per cent were found to be of such low dollar value that they were ordered in large quantities at infrequent intervals. This fact would invalidate the demand data to the extent that it would not reveal the true demand distribution. In the remaining items, 27 per cent were found to have inadequate demand for a valid Chi-square test. Therefore, based on the results of the previous analysis, the theoretical Poisson distribution was selected to approximate the demand distribution for the class of patent medicines considered in this study.

Lead-time Distribution Analysis

On Monday morning of each week a group order is sent to the central warehouse for all items entered in the weekly order book during the week. The items included in this book are delivered to the store by the afternoon of the same day. It should be reasonable to assume that fewer items would need replenishing shortly after the receipt of the Monday order than in the latter part of the week. Thus, the ordering frequency for various items during the week should be distributed as shown in Figure 5.

Since an order for an individual item is considered to be placed when it is entered in the weekly order book, the ordering frequency and lead-time will have the same distribution. The theoretical Poisson distribution closely resenbles the distribution shown in Figure 5 for small values of the mean. Therefore, based on the assumption that replenishment orders are entered in the order book at discrete random intervals,

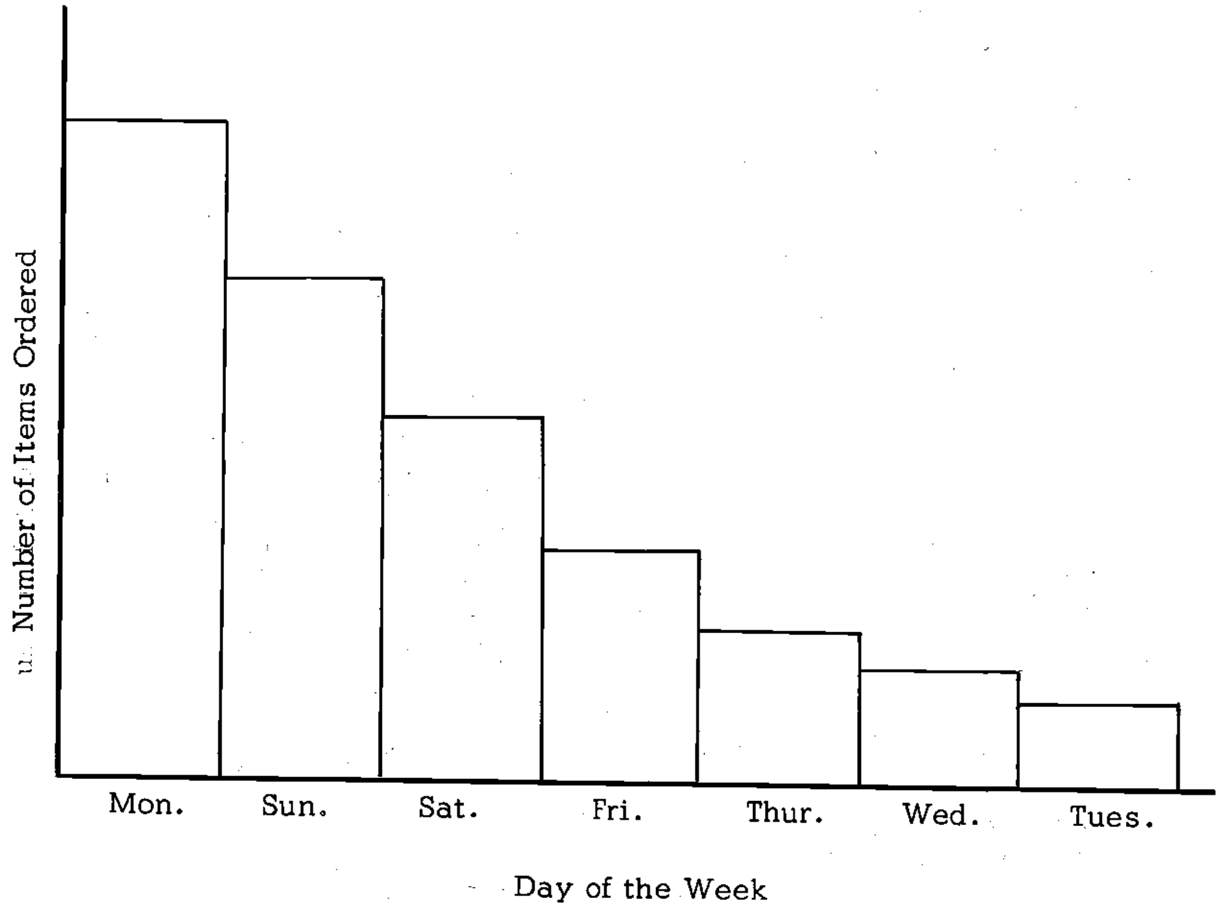


Figure 5. Hypothetical Representation of the Distribution of Replenishment Orders During a Week.

the lead-time will have discrete random lengths of duration. On the further assumption that the lead-time is independent of order size, the Poisson distribution was selected to approximate the lead-time variable.

<u>Safety Inventory Model</u>

The reorder point equation for the combined function of a Poisson distributed demand and Poisson distributed lead-time is as follows:[7]

for d = the random variable of demand per unit time

x = the lead-time random variable

$D(x)$ = the demand random variable for a lead-time of x units

μ_d = the mean value of (d)

μ_x = the mean value of (x)

$P(S)^*$ = the probability of a stock-out during the replenishment lead-time

RP = the reorder point inventory level

$$P(S)^* = \sum_{x=0}^{7} \frac{e^{-\mu_x}(\mu_x)^x}{x!} \sum_{D(x)=RP+1}^{\infty} \frac{e^{-x\mu_d}(x\mu_d)^{D(x)}}{D(x)!} \qquad (14)$$

The above mathematical relation is stated in discrete terms because

---

[7] Ekey, Talbird and Newberry, <u>op</u>. <u>cit.</u>, p. 33.

demand and replenishment lead-time are assumed to be discrete functions with lead-time being bounded with an upper value of seven days. Values for $P(S)^*$ in Equation (14) were calculated for μ_d equal to 0.25, 0.50, 0.75, 1.00 and 2.00 units per day. The results of these calculations were presented graphically in Figure 6 on coordinates of probability of a stock-out during lead-time versus reorder point inventory level. Since the inventory is composed of discrete items, partial items were not permitted and the five values of μ_d were sufficient to determine the reorder point inventory level for each item in the group of items under study at various levels of probability of a stock-out.

In order to maintain the aggregate characteristics of the inventory model, a procedure was needed which would distribute the safety inventory investment and protection level among the individual items based on a single policy as established by management for the class of items. The desired distribution among the individual items as described in Chapter III, "Safety Inventory Investment Distribution" was expressed as follows:

$$P(S_i) = \overline{P(S)} \sqrt{\frac{C_i}{\overline{C}}} \qquad (15)$$

where:

$P(S_i)$ = the probability of one or more stock-outs of the i^{th} item during a period

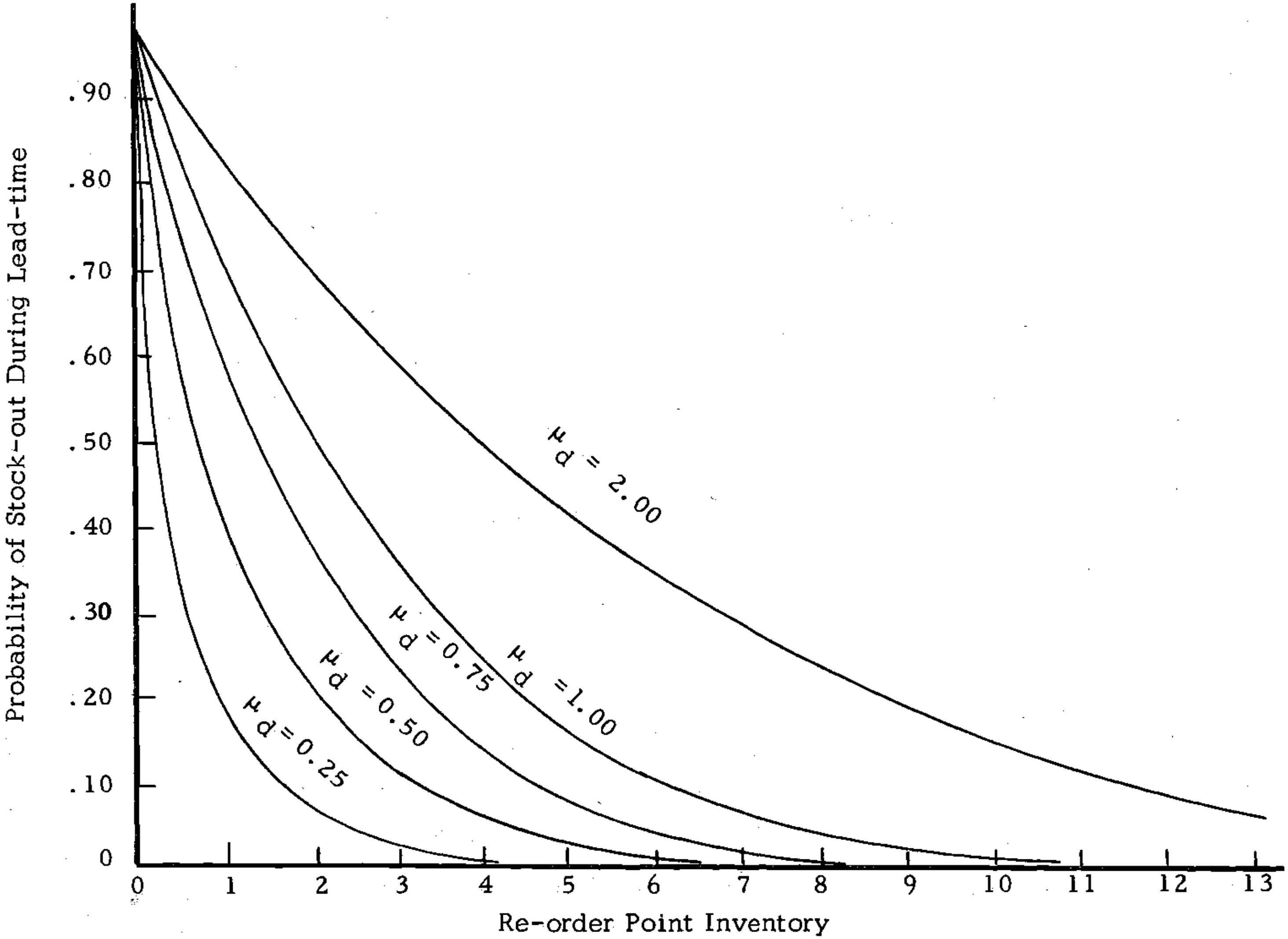


Figure 6. Graph of the Probability of a Stock-out During Lead Time and Re-order Point Inventory Level for a Poisson Demand Function with Mean (μ_d) and a Poisson Lead Time Function with Mean Constant at $\mu_X = 3$ Days.

$\overline{P(S)}$ = the probability of one or more stock-outs for the class of item during a period

C_i = the unit cost of the i^{th} item

$\overline{C}$ = the average unit cost of the class of items

Equation (15) was evaluated for each of the items under study for $\overline{P(S)} = 0.04$ 0.10, 0.20, 0.40 and 0.60. These values were selected in order to satisfactorily cover the range of policies which the store manager would likely consider for application.

In order to determine the reorder point inventory level for each item corresponding to the service level policy established by management ($\overline{P(S)}$), it was necessary to convert the probability of one or more stock-outs of the i^{th} item during a period to the probability of one or more stock-outs of the i^{th} item during a cycle. This conversion, which would facilitate the use of Figure 6 in determining the reorder point for each item, was made by Equation (3)[8] as follows:

$$P(S_i)^* = 1 - [1 - P(S_i)]^{\frac{1}{n_i}} \qquad (16)$$

where n_i = the number of times the i^{th} item will be ordered during a period or the number of cycles per period.

*Appendix B.

$P(S_i)^*$ = the probability of one or more stock-outs for the i^{th} item during a cycle.

Equation (16) was evaluated for each item at three levels of total number of replenishment orders each period (i.e., $\sum_{i=1}^{138} n_i$ = 300, 400 and 600) and the values for $P(S_i)$ at five levels of $\overline{P(S)}$ calculated in Equation (15).

The reorder point for each item was determined by referring the values calculated in Equation (16) and the corresponding average usage per day value (μ_d) for each item to the graph of Figure 6.

Safety Inventory Investment

In order to determine the safety inventory investment for each item, the average usage during lead-time was subtracted from the reorder point inventory level. The safety inventory investment for each item was calculated at the five levels for probability of a stock for the class of items. The aggregate investment in safety stock was calculated as the total investment in the individual items.

The aggregate safety inventory investment for the total number of replenishment orders per period equal to 300, 400 and 600 were graphed on coordinates of probability of a stock-out for the class of items versus total dollar investment in safety inventory and presented in Figure 7.

Combined Cycle and Safety Inventory Investment

In order to provide the store management with complete knowledge

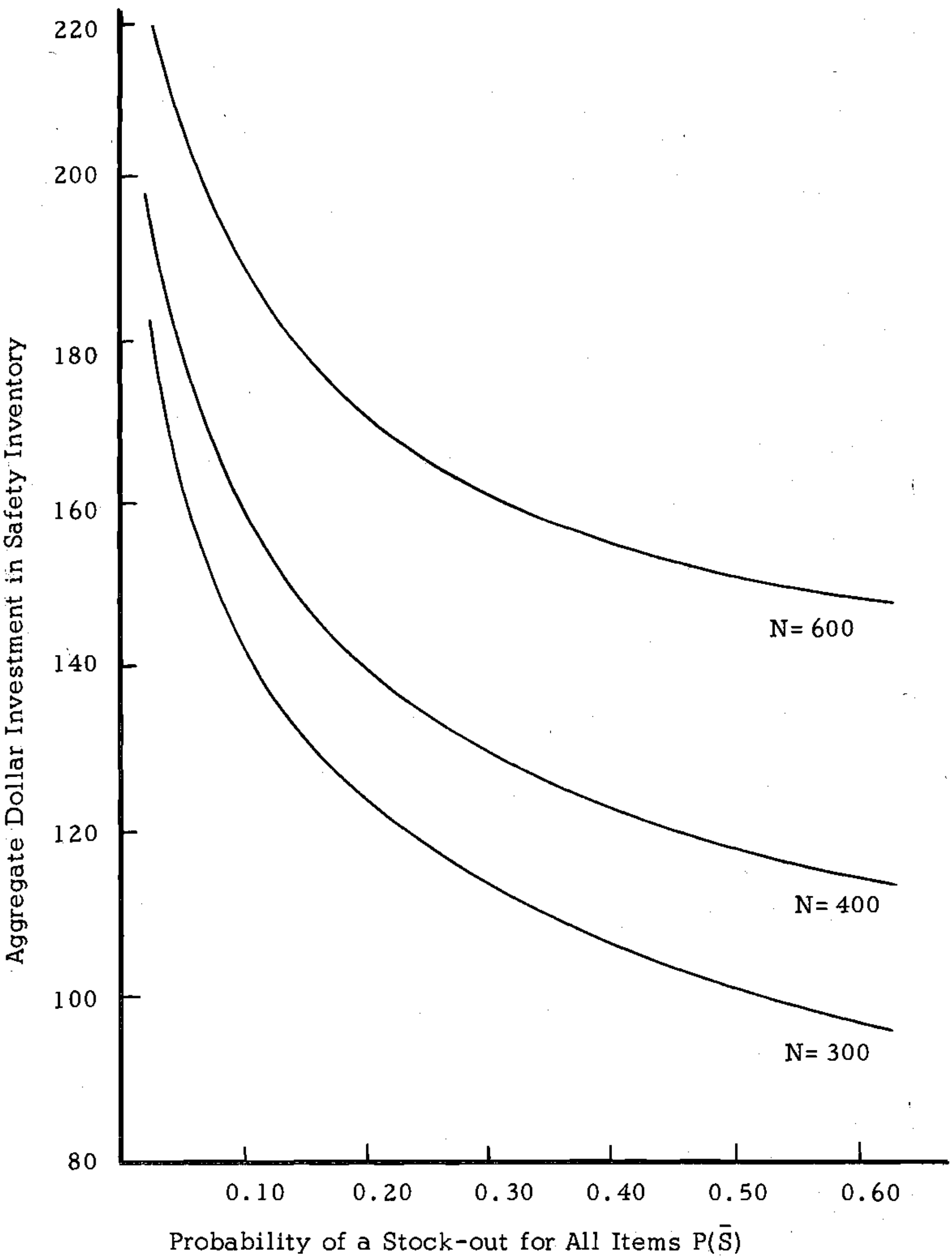


Figure 7. Graph of the Total Dollar Investment in Safety Inventory Versus the Probability of a Stock-out for the Class of Items at Three Levels of Total Number of Replenishment Each Period.

of the interrelationships between probability of a stock-out, ordering frequency and total (cycle and safety) inventory investment, the graphs of Figures 3 and 7 were combined into one graph and presented in Figure 8. The coordinates of this graph are average aggregate dollar investment in inventory versus probability of a stock-out for the class of items. The curves of Figure 8 were constructed by adding the average cycle inventory investment at given levels of total number of replenishment orders to the corresponding safety inventory curve for the various levels of probability of a stock-out. The total inventory curves of Figure 8 are terminated at probability of a stock-out equal to 0.60 because it is unlikely that a policy with the probability greater than 0.60 would be considered by the store management.

Figure 8 represents a continuous series of inventory policies which displays particular emphasis on the 95 per cent confidence interval containing the policy that minimizes the total cost relation.

Inventory Policy Selection

The graph of Figure 8 was presented to the store manager for his review. The meaning of each of the terms on the graph were explained in lay language and an example of how an inventory policy could be selected was illustrated. The conversation continued until it was considered that the store manager fully comprehended the meaning and interrelationship of

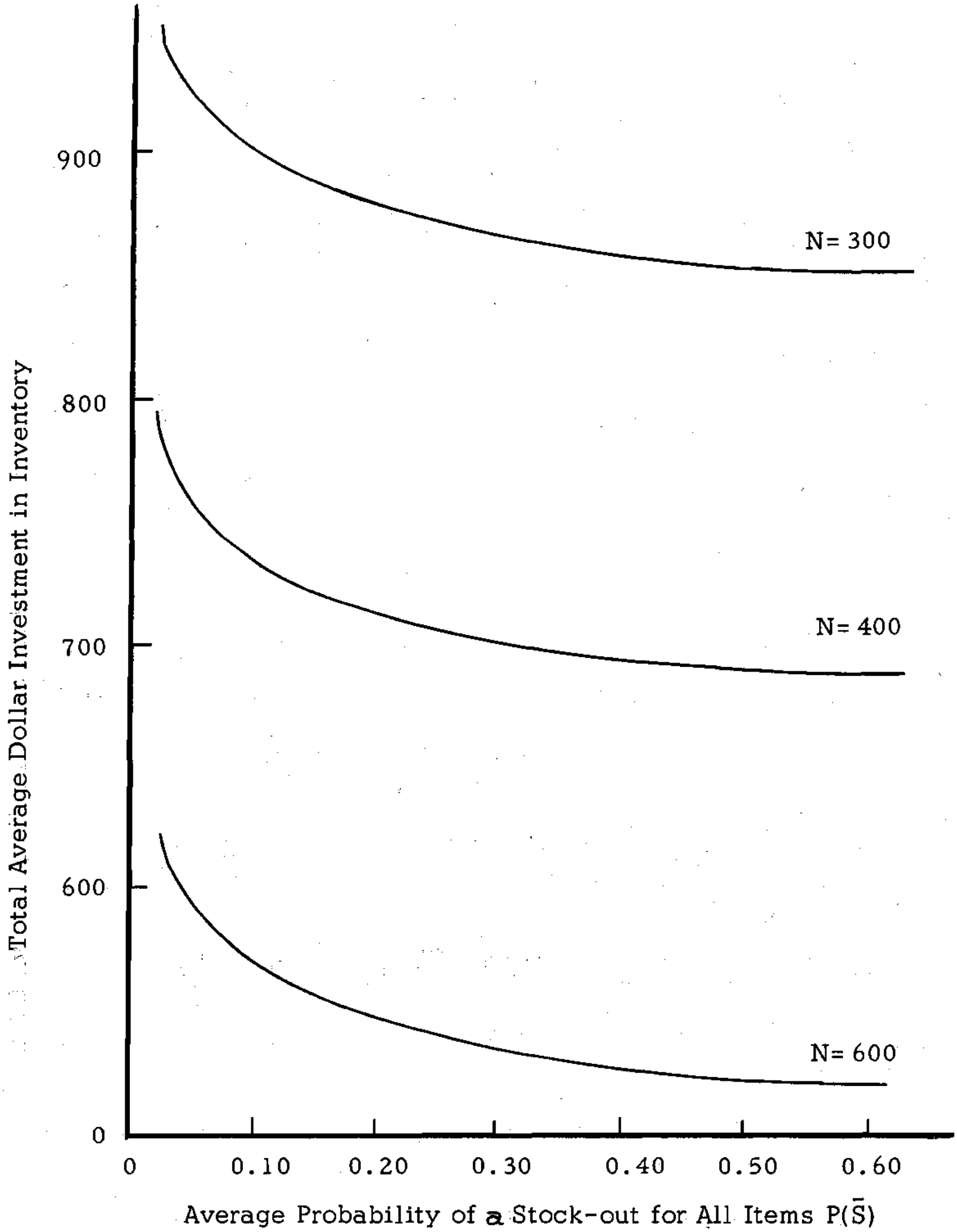


Figure 8. Graph of the Total Dollar Investment in Inventory Versus Average Probability of a Stock-out for All Items at Three Levels of Total Number of Replenishment Orders Each Period.

the factors contained on the graph of Figure 8. The store manager was then asked to select an inventory policy from the graph which would best fill his management objectives for the store. The store manager selected a probability of stock-out for the class of drugs equal to 0.10 and the total number of replenishment orders during a period equal to 400. The corresponding total investment in inventory was calculated to be $731.92. This total was broken down to have $574.60 invested in cycle inventory and $157.32 invested in safety inventory.

Presentation of the Selected Policy

The order quantity and reorder point for each of the items in the sample were calculated by the same analysis used to develop the curves in the graph of Figure 8. The order quantity and reorder point for each drug item under the aggregate policy selected by the store management are presented in Table 4 of Appendix E.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

During the course of investigation and analysis which has led to the completion of this study, certain facts concerning inventory control were confirmed, while other areas, outside the scope of this study, were found deserving of further study. Based upon these findings, the conclusions and recommendations for future study are presented.

Before presenting the conclusions and recommendations, the following qualifications of the study are presented:

1. This study is based on the inventory records of one retail drug store.
2. This study is based upon an inventory model utilizing one economic lot size formula and one reorder point formula.
3. This study is based upon the sales and cost characteristics of patent medicines.

Conclusions

From the inventory model and information developed in this study, the following conclusions may be drawn:

1. In developing purchasing policies for an aggregate of drug items, consideration should be given to factors other than inventory investment. Specifically, managements objectives, financial resources and facility limitations must be considered.
2. A statistical analysis of the variable cost factors, cost to place replenishment orders and cost to maintain inventory, can be used to compute a confidence interval within which is contained the cycle inventory policy that minimizes the equation of total relevant inventory cost. Such a range of cycle inventory policies is defined by a maximum and minimum value for cycle inventory investment and total annual replenishment orders.
3. The sales activity of patent medicines is such that an aggregate of items can be grouped together for aggregate inventory control.
4. The inventory records of retail drug stores may lack the information necessary for the application of this model.
5. The model developed in this study provides the management of a retail drug store with useful and reliable guides for setting purchasing policies on an aggregate of items.

Recommendations

For the purpose of extending knowledge in the area of aggregate inventory control, the following recommendations are made with respect to further studies:

1. Studies should be undertaken to ascertain the sensitivity of aggregate inventory policies to total relevant cost.
2. Sets of reorder point tables and charts should be developed from the joint density function of various theoretical probability distributions over a wide range of values for demand and lead-time.
3. Methods for accurately estimating the ordering costs and inventory carrying costs should be investigated for various typical situations.
4. Studies should be undertaken to ascertain which demand and lead-time distributions are most applicable to typical industrial situations.
5. Study should be devoted to the development of a computer analog for the analysis of the effects relevant cost factors have upon aggregate inventory policies.

APPENDICES

APPENDIX A

DERIVATION OF THE OPTIMUM ORDER QUANTITY RELATIONS

The order quantity equation that minimizes the total relevant cost function is derived as follows:

for D_i = The period demand for the i^{th} item in units.

Q_i = The quantity of the i^{th} item ordered in each replenishment order for the i^{th} item (units).

S = The cost to place a replenishment order (dollars).

C_i = The unit cost of the i^{th} item (dollars).

r = The per cent of unit cost representing the cost to maintain an item in inventory for one period.

T = The total relevant cost for maintaining and replenishing inventory.

n_i = The number of replenishment orders each period for the i^{th} item.

A_i = The average dollar investment in cycle inventory for the i^{th} item.

T = REPLENISHMENT COST + CARRYING COST[1]

---

[1] Course Notes -(Materials Control) I.E. 606. Prof. E.C. Franklin, Assoc. Proffessor of Industrial Engineering, Georgia Institute of Technology, 1962. See also Miller and Starr (11).

$$T_i = \frac{D_i}{Q_i} S + \frac{Q_i}{2} rC_i \tag{A1}$$

Differentiating Equation (A-1) with respect to reorder quantity (dQ):

$$\frac{dT_i}{dQ_i} = -D_i S Q_i^{-2} + \frac{rC_i}{2} .$$

Solving for the Q_i which minimizes the total relevant cost relation:

$$Q_i = \sqrt{\frac{2D_i S}{rC_i}} . \tag{A2}$$

The relation for the total number of replenishment orders is:

$$\sum_i N_i = \sum_i \frac{D_i}{Q_i} . \tag{A3}$$

Substituting the relation for the replenishment order quantity (A2) into Equation (A3):

$$\sum_i N_i = \sum_i \frac{D_i}{\sqrt{\frac{2D_i S}{rC_i}}} = \sum_i \sqrt{\frac{D_i C_i r}{2S}} = \sqrt{\frac{r}{2s}} \sum_i \sqrt{D_i C_i} \tag{A4}$$

The relation for the average dollar investment in working inventory is:

$$\sum_i A_i = \sum_i \frac{Q_i C_i}{2} \tag{A5}$$

Substituting the relation for the replenishment order quantity (A2) into Equation (A5):

$$\sum_i A_i = \sum_i \sqrt{\frac{2D_i S}{rC_i}} \left(\frac{C_i}{2} \right) = \sqrt{\frac{S}{2r}} \sum_i \sqrt{D_i C_i} \tag{A6}$$

Multiplying the expression for the average dollar investment in working inventory (A6) and the expression for the total number of replenishment orders (A4):

$$\sum_i (A_i)(N_i) = \left(\sqrt{\frac{s}{2r}} \sum_i \sqrt{D_i C_i} \right) \left(\sqrt{\frac{r}{2s}} \sum_i \sqrt{D_i C_i} \right) \tag{A7}$$

$$= \frac{1}{2} \left(\sum_i \sqrt{D_i C_i} \right)^2$$

APPENDIX B

DERIVATION OF THE PROBABILITY OF ONE OR MORE STOCK-OUTS DURING LEAD-TIME

The equation which expresses the probability of one or more stock-outs during lead-time is as follows:

for $P(S_i)$ = the probability of one or more stock-outs for the i^{th} item during a period (n_i cycles).

$P(S_i)^*$ = the probability of one or more stock-outs for the i^{th} item during any one lead-time.

n_i = the number of replenishment orders placed during a given period for the i^{th} item (i.e., the number of replenishment order cycles during a given period).

Then

$$\begin{bmatrix} \text{The probability of no stock-out} \\ \text{for the } i^{th} \text{ item during any one} \\ \text{cycle} \end{bmatrix} = [1 - P(S_i)^*] \qquad \text{(B1)}$$

and

$$\begin{bmatrix} \text{The probability of no stock-outs} \\ \text{for the } i^{th} \text{ item during } n_i \text{ cycles} \end{bmatrix} = [1 - P(S_i)^*]^{n_i} \qquad \text{(B2)}$$

Following from Equation (B2) and the definition of symbols above:

$$P(S_i) = 1 - [1 - P(S_i)^*]^{n_i} \tag{B3}$$

Simplifying Equation (B3) as follows:

$$1 - P(S_i) = [1 - P(S_i)^*]^{n_i} \tag{B4}$$

and

$$[1 - P(S_i)]^{\frac{1}{n_i}} = 1 - P(S_i)^* \tag{B5}$$

The following relation is derived by transposing the terms of Equation (B5):

$$P(S_i)^* = 1 - [1 - P(S_i)]^{\frac{1}{n_i}} \tag{B6}$$

APPENDIX C

DERIVATION OF THE EQUATION FOR THE EXPECTED VALUE AND VARIANCE OF THE RATIO OF TWO RANDOM VARIABLES[1]

Let $X_1, X_2, \ldots, X_n$ be independent random variables with means at $\mu_1, \mu_2, \ldots, \mu_n$, respectively and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively. Suppose the X's are related by

$$f(X_1, X_2, \ldots, X_n)$$

then the expected value

$$E[\, f(X_1, X_2, \ldots, X_n)] \cong f(\mu_1, \mu_2, \ldots, \mu_n) \qquad \text{(C1)}$$

and the variance

$$\text{var. } [\, f(X_1, X_2, \ldots, X_n)] = \sigma_f^2 \qquad \text{(C2)}$$

$$\cong \sigma_1^2 \left(\frac{\partial f}{\partial X_1} \bigg|_{\mu_1 \ldots \mu_n} \right)^2 + \ldots + \sigma_n^2 \left(\frac{\partial f}{\partial X_n} \bigg|_{\mu_1 \ldots \mu_n} \right)^2$$

If the X's are normally distributed, $f(X_1, X_2, \ldots, X_n)$ is approximately normally distributed.

---

[1] Course notes - Industrial Engineering 439. Lynwood A. Johnson, Assistant Professor of Industrial Engineering, Georgia Institute of Technology.

Derivation for Single Variable Function

In a Taylor's series for a single variable function f(X):

$$f(X+h) = f(X) + hf'(X) + \frac{h^2}{2!} f''(X) + \ldots \tag{C3}$$

consider any measurement X_i as being made up of two components, one fixed and the other variable. Thus, X_i can be expressed as follows:

$$X_i = \mu_i + \epsilon_i \tag{C4}$$

where

$$E(\epsilon_i) = 0 \text{ and var. } (\epsilon_i) = \sigma_1^2$$

further:

$$E(X_i) = E(\mu_i + \epsilon_i) = E(\mu_i) + E(\epsilon_i) = \mu_i + 0 = \mu_i \tag{C5}$$

$$\text{var. } (X_i) = \text{var.}(\mu_i + \epsilon_i) = \text{var.}(\mu_i) + \text{var.}(\epsilon_i) = 0 + \sigma_1^2 = \sigma_1^2 \tag{C6}$$

Derivation for Multiple Variable Function

In a Taylor's series for a multiple variable function $f(X_1, X_2, \ldots, X_n)$:

for $$f(X_1, X_2, \ldots, X_n) = f(X_1 + \epsilon_1, X_2 + \epsilon_2, \ldots, X_n + \epsilon_n)$$

then,

$$f(X_1, X_2, \ldots, X_n) = f(\mu_1, \mu_2, \ldots, \mu_n) + \epsilon_1 \frac{\partial f}{\partial X_1}\bigg|_{\mu_1, \ldots, \mu_n} \tag{C7}$$

$$+ \epsilon_2 \frac{\partial f}{\partial X_2}\bigg|_{\mu_1 \ldots \mu_n} + \ldots + \epsilon_n \frac{\partial f}{\partial X_n}\bigg|_{\mu_1, \ldots, \mu_n}$$

$$+ \frac{\epsilon_1^2}{2!} \frac{\partial^2 f}{\partial x_1^2}\bigg|_{\mu_1, \ldots, \mu_n} + \frac{\epsilon_n^2}{2!} \frac{\partial^2 f}{\partial X_n}\bigg|_{\mu_1, \ldots, \mu_n} + \ldots$$

Neglecting all terms of order higher than ϵ_i and the theorems on linear combination:[2]

since $E(\epsilon_i) = 0$, for all i,

$$E(f) \cong f(\mu_1, \mu_2, \ldots, \mu_n) \tag{C8}$$

and

$$\text{var.}(f) \cong 0 + \text{var.}\left(\frac{\partial f}{\partial X_1}\bigg|_{\mu_1, \ldots, \mu_n}\right)^2 + \text{var.}(\epsilon_2)\left(\frac{\partial f}{\partial X_2}\bigg|_{\mu_1, \ldots, \mu_n}\right)^2$$

$$+ \ldots + \text{var.}(\epsilon_n)\left(\frac{\partial f}{\partial X_n}\bigg|_{\mu_1, \ldots, \mu_n}\right)^2 \tag{C9}$$

$$\text{var.}(f) \cong \sigma_1^2\left(\frac{\partial f}{\partial X_1}\bigg|_{\mu_1, \ldots, \mu_n}\right)^2 + \sigma_2^2\left(\frac{\partial f}{\partial X_2}\bigg|_{\mu_1, \ldots, \mu_n}\right)^2$$

$$+ \ldots + \sigma_n\left(\frac{\partial f}{\partial X_n}\bigg|_{\mu_1, \ldots, \mu_n}\right) \tag{C10}$$

[2]Bowker, A. H. and Lieberman, G. J., Engineering Statistics, Prentice Hall, Inc., Englewood Cliffs, N. J., 1959.

Application to the Ratio of Two Normally Distributed Random Variables

In a function containing two variables X_1, and X_2:

$$Y = f(X_1, X_2) = \frac{X_1}{X_2} \tag{C11}$$

the expected value of the bi-variable function corresponding to Equation (C8) is as follows:

$$E(X_1/X_2) \cong \mu_1/\mu_2 \tag{C12}$$

The variance of the bi-variable function corresponding to Equation (C10) is as follows:

for $$\frac{\partial f}{\partial X_1} = \frac{1}{X_2} \; ; \; \left.\frac{\partial f}{\partial X_1}\right|_{\mu_1,\mu_2} = \frac{1}{\mu_2}$$

and $$\frac{\partial f}{\partial X_2} = -\frac{X_1}{X_2} \; ; \; \frac{\partial f}{\partial X_2} = -\frac{\mu_1}{\mu_2}$$

therefore

$$\sigma_Y^2 = \sigma_1^2\left(\frac{1}{\mu_2}\right)^2 + \sigma_2^2\left(\frac{\mu_1^2}{\mu_2^2}\right) = \frac{\mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2}{\mu_2^4} \tag{C13}$$

APPENDIX D

CHI-SQUARE TEST

An example of the Chi-square test[1] is provided for the purpose of illustrating the procedure used in testing the demand distribution for the class of drugs under study to (1) the theoretical normal distribution and (2) the theoretical Poisson distribution.

Test Example

The test of item number 49 was arbitrarily selected for the illustration. The test was begun with the statement of a hypothesis concerning each of the theoretical distributions.

Hypothesis for the Normal Distribution

"The demand distribution for item number 49 is from a normally distributed population." The Chi-square test of this hypothesis is as follows:

1. Demand Data

Week	Demand	Week	Demand
1	4	5	3
2	0	6	0
3	2	7	4
4	0	8	0

[1] Bowker and Lieberman, op. cit., p. 366.

Demand Data (Continued)

Week	Demand	Week	Demand
9	1	18	4
10	12	19	0
11	0	20	15
12	3	21	0
13	0	22	1
14	3	23	0
15	1	24	1
16	0	25	0
17	3	26	3

Mean Demand = 2.46 Units/Weeks

Standard Deviation of Demand = 2.60

2. Test Statistic

$$X^2 = \sum_{i=1}^{k} \frac{(O_i - NE_i)^2}{NE_i}$$

where K = the number of categories

O_i = the number of actual observations of demand in the i^{th} category

NE_i = the normal expected number of observations in the i^{th} category.

3. Test

Category	Demand Range	O_i	NE_i	X^2
1.	0 - 1	15	7	9.14
2.	2 - 3	6	8	0.50
3.	4 - 5	3	6	1.50
4.	6 - ∞	2	3	0.33
				11.47

4. Decision

Degrees of freedom = 3(i.e., k-1)

Level of significance = 0.05

$$X^2_{0.05,3} = 7.82$$

$$11.47 > X^2_{0.05,3} = 7.82$$

therefore, the hypothesis is rejected.

Hypothesis for the Poisson Distribution

"The demand distribution for the item number 49 is from a Poisson distributed population." The Chi-square test of this hypothesis is as follows:

1. the data is the same as presented in the previous test;

2. test statistic:

$$X^2 = \sum_{i=1}^{k} \frac{(O_i - PE_i)^2}{PE_i}$$

where PE_i = the Poisson expected number of observations in the i^{th} category.

3. Test

Category	Demand Range	O_i	PE_i	X_i^2
1.	0 - 1	15	14	0.07
2.	2 - 3	6	9	1.00
3.	4 - 5	3	2	0.50
4.	6 - ∞	2	1	1.00
				2.57

4. Decision

Degrees of freedom = 3(i.e., k-1)

Level of significance = 0.05

$$X^2_{0.05,3} = 7.82$$

$$2.57 < X^2_{0.05,3} = 7.82$$

therefore, the hypothesis is accepted.

APPENDIX E

DATA

Table 2. Unit Cost and Demand Data for the Drug Items Under Study

Item	Unit Cost	Annual Dollar Demand	Cumulative Dollar Demand	Cumulative Per Cent of Total Dollar Demand
1	0.57	143.64	143.64	3.5
2	0.84	124.32	267.96	6.5
3	0.34	116.28	384.24	9.4
4	0.90	100.80	485.04	11.9
5	1.81	97.74	582.78	14.3
6	0.39	95.16	677.94	16.6
7	0.58	93.96	771.90	19.0
8	1.49	86.42	858.32	21.1
9	0.29	82.94	941.26	23.5
10	0.40	82.40	1023.66	25.2
11	0.32	81.28	1104.94	27.2
12	0.45	73.80	1178.74	29.1
13	0.18	73.44	1252.18	30.9
14	0.17	72.08	1324.26	32.6
15	0.15	72.00	1378.26	34.4
16	0.36	69.12	1465.38	36.1
17	0.26	68.64	1534.02	37.8
18	0.24	67.20	1601.22	39.8
19	1.20	67.20	1668.42	41.2
20	0.19	66.12	1734.54	42.3
21	0.56	59.36	1793.90	44.1
22	0.56	58.24	1852.14	45.6
23	0.43	56.76	1908.90	47.1
24	0.29	55.68	1964.58	48.5
25	0.24	55.68	2020.26	49.8
26	0.56	50.40	2070.66	51.0
27	0.36	46.80	2117.46	52.0
28	0.46	46.00	2163.46	53.0
29	0.89	44.50	2207.96	54.3
30	1.67	43.42	2251.38	55.4
31	0.35	43.40	2294.74	56.5
32	0.70	39.20	2333.98	57.4
33	0.22	36.08	2370.06	58.4
34	0.42	35.28	2405.34	59.2
35	0.14	34.72	2440.06	59.6

Table 2. Unit Cost and Demand Data for the Drug Items Under Study (Continued)

Item	Unit Cost	Annual Dollar Demand	Cumulative Dollar Demand	Cumulative Per Cent of Total Dollar Demand
36	0.43	34.40	2474.46	60.9
37	0.40	34.40	2508.86	62.0
38	0.19	34.20	2543.06	62.8
39	1.20	33.60	2576.66	63.4
40	0.66	33.00	2609.66	64.2
41	0.37	32.56	2642.22	65.2
42	0.05	32.40	2674.62	66.0
43	0.38	31.92	2706.54	66.8
44	0.36	31.68	2738.22	67.5
45	1.11	31.08	2769.30	68.2
46	0.37	29.60	2798.90	68.9
47	0.64	29.42	2828.34	69.6
48	0.42	28.56	2856.90	70.5
49	0.22	28.16	2885.06	71.2
50	0.40	27.20	2912.26	71.8
51	0.50	27.00	2939.26	72.4
52	0.55	26.40	2965.66	73.0
53	0.72	25.92	2991.58	73.7
54	0.61	25.62	3017.20	74.4
55	0.20	24.80	3042.00	75.0
56	0.28	24.08	3066.08	75.7
57	0.20	23.60	3089.64	76.2
58	0.63	22.68	3112.36	76.8
59	0.24	22.56	3135.92	77.5
60	1.11	22.20	3157.12	78.2
61	0.34	21.76	3178.88	78.7
62	0.45	21.60	3200.48	79.0
63	0.51	21.42	3221.90	79.5
64	0.56	21.28	3243.18	80.0
65	0.59	21.24	3264.42	80.5
66	0.59	21.24	3285.66	81.0
67	0.22	21.12	3306.78	81.5
68	0.81	21.06	3327.84	82.0
69	0.72	20.16	3348.00	82.5

Table 2. Unit Cost and Demand for the Drug Items Under Study (Continued)

Item	Unit Cost	Annual Dollar Demand	Cumulative Dollar Demand	Cumulative Per Cent of Total Dollar Demand
70	0.31	19.84	3367.84	83.0
71	0.62	19.84	3387.68	83.5
72	0.34	19.04	3406.72	84.0
73	0.63	18.90	3425.62	84.5
74	0.19	18.24	3443.86	85.0
75	0.70	18.20	3462.06	85.5
76	0.23	17.94	3480.00	86.0
77	0.69	17.94	3497.94	86.4
78	1.11	17.76	3515.70	86.8
79	0.54	17.28	3532.98	87.2
80	0.22	16.72	3549.70	87.6
81	0.57	15.96	3565.66	88.0
82	0.34	15.64	3581.30	88.4
83	0.60	15.60	3596.90	88.8
84	0.33	15.18	3612.08	89.2
85	0.75	15.00	3627.08	89.6
86	0.67	14.74	3641.82	90.0
87	0.81	14.58	3656.40	90.4
88	0.34	14.28	3670.68	90.8
89	0.84	13.44	3684.12	91.1
90	0.41	13.12	3697.24	91.4
91	0.38	13.92	3710.16	91.7
92	0.17	13.92	3723.08	92.0
93	0.06	12.72	3735.80	92.3
94	0.25	12.50	3748.30	92.6
95	0.26	12.48	3760.78	92.9
96	0.87	12.18	3772.96	93.2
97	0.42	11.76	3784.72	93.5
98	0.48	11.52	3796.24	93.8
99	0.48	11.52	3807.76	94.1
100	0.71	11.36	3819.12	94.4
101	1.13	11.30	3830.42	94.7
102	0.20	11.20	3841.62	94.9
103	0.62	11.16	3852.78	95.1

Table 2. Unit Cost and Demand for the Drug Items Under Study (Continued)

Item	Unit Cost	Annual Dollar Demand	Cumulative Dollar Demand	Cumulative Per Cent of Total Dollar Demand
104	0.34	10.88	3863.66	95.3
105	0.54	10.80	3874.46	95.5
106	1.32	10.56	3885.02	95.7
107	0.44	10.56	3895.88	95.9
108	0.12	10.32	3905.90	96.2
109	1.28	10.24	3916.14	96.4
110	0.71	9.94	3926.08	96.6
111	0.56	8.96	3935.04	96.9
112	0.73	8.76	3942.80	97.1
113	2.00	8.00	3951.80	97.4
114	0.56	7.84	3959.64	97.6
115	0.26	7.80	3967.44	97.8
116	0.43	7.74	3975.18	98.0
117	0.75	7.50	3982.68	98.2
118	0.16	7.04	3989.72	98.4
119	0.42	6.72	3996.44	98.6
120	0.15	6.60	4002.04	98.7
121	0.50	6.00	4008.04	98.8
122	0.96	5.76	4014.80	98.9
123	0.12	5.76	4020.56	99.1
124	0.71	5.68	4026.24	99.2
125	0.80	5.28	2031.34	99.3
126	1.06	4.24	4035.76	99.4
127	0.41	4.10	4039.86	99.5
128	0.68	4.08	2043.94	99.6
129	0.63	3.78	4047.72	99.6
130	0.09	3.24	4050.96	99.7
131	0.51	3.06	4054.02	99.7
132	0.46	2.76	4056.78	99.8
133	0.25	2.50	4059.28	99.8
134	0.27	2.16	4061.44	99.8
135	0.50	2.00	4063.44	99.9
136	0.28	1.68	4065.12	99.9
137	0.20	1.20	4066.32	99.9
138	0.23	.92	4067.24	100.0

Table 3. The Results of the Chi-Square Test of the Hypothesis that the Demand Distribution of the i^{th} Item is from (Case I.) a Normally Distributed Population, (Case II.) a Poisson Distributed Population

Actual Chi-Square Values for the Test of the Hypothesis that the Demand Distribution of the i^{th} Item is from a

Item Number	Normally Distributed Population		Poisson Distributed Population		Chi-Square Values for the 5% Confidence Level
1	4.70	Accept	1.30	Accept	7.82
3	7.50	Reject	31.30	Reject	5.99
4	33.70	Reject	12.78	Reject	5.99
5	18.70	Reject	3.80	Accept	7.82
7	5.60	Accept	3.85	Accept	5.99
8	8.30	Reject	4.19	Accept	5.99
9	2.24	Accept	3.99	Accept	7.82
10	3.60	Accept	3.61	Accept	5.99
12	2.78	Accept	2.17	Accept	7.82
14	7.30	Reject	5.56	Accept	5.99
15	vh	Reject	vh	Reject	5.99
17	2.50	Accept	4.18	Accept	7.82
18	10.60	Reject	15.88	Reject	7.82
19	24.90	Reject	1.14	Accept	7.82
21	18.10	Reject	0.24	Accept	5.99
23	9.10	Reject	4.67	Accept	7.82
24	vh	Reject	vh	Reject	7.82
25	6.90	Reject	0.19	Accept	5.99
28	7.30	Reject	4.18	Accept	5.99
30	19.70	Reject	2.36	Accept	5.99
31	14.20	Reject	23.40	Reject	7.82
32	22.60	Reject	30.15	Reject	5.99
34	2.80	Accept	4.20	Accept	5.99
37	1.20	Accept	1.85	Accept	5.99
40	18.30	Reject	1.17	Accept	5.99
41	12.50	Reject	4.67	Accept	5.99
42	vh	Reject	vh	Reject	7.82
44	7.70	Reject	1.98	Accept	5.99

Table 3. The Results of the Chi-Square Test of the Hypothesis that the Demand Distribution of the i^{th} Item is from (Case I.) a Normally Distributed Population, (Case II.) a Poisson Distributed Population

Item Number	Actual Chi-Square Values for the Test of the Hypothesis that the Demand Distribution of the i^{th} Item is from a Normally Distributed Population		Poisson Distributed Population		Chi-Square Values for the 5% Confidence Level
46	3.85	Accept	0.06	Accept	5.99
47	11.10	Reject	2.30	Accept	5.99
48	13.60	Reject	1.30	Accept	7.82
49	11.47	Reject	2.57	Accept	7.82
50	8.80	Reject	1.01	Accept	5.99
51	16.00	Reject	0.22	Accept	5.99
53	33.70	Reject	14.80	Reject	7.82
55	13.20	Reject	7.77	Reject	5.99
56	6.73	Reject	0.87	Accept	5.99
57	6.50	Reject	4.45	Accept	5.99
59	9.60	Reject	4.20	Accept	7.82
61	12.70	Reject	4.50	Accept	7.82
62	24.90	Reject	1.40	Accept	7.82
68	19.70	Reject	2.63	Accept	7.82
70	15.70	Reject	5.16	Accept	5.99
72	5.70	Accept	0.20	Accept	5.99
73	10.10	Reject	0.00	Accept	5.99
75	19.70	Reject	2.63	Accept	7.82
76	16.00	Reject	2.45	Accept	7.82
79	34.70	Reject	0.38	Accept	7.82
80	8.30	Reject	4.00	Accept	5.99
88	5.00	Accept	1.45	Accept	7.82
90	7.78	Reject	0.18	Accept	5.99
105	50.90	Reject	4.10	Accept	5.99
108	7.10	Reject	4.67	Accept	5.99
120	24.90	Reject	1.40	Accept	5.99
130	36.70	Reject	5.56	Accept	5.99

Table 4. Order Quantity and Reorder Point for Each Drug Item Under the Selected Aggregate Policy

Item	Order Quantity	Reorder Point	Item	Order Quantity	Reorder Point
1	36	8	27	33	5
2	23	6	28	25	5
3	56	11	29	14	3
4	19	6	30	8	3
5	9	2	31	31	5
6	54	6	32	15	3
7	28	5	33	48	6
8	11	3	34	24	6
9	54	6	35	72	9
10	39	7	36	24	5
11	49	7	37	25	5
12	32	6	38	54	8
13	82	11	39	7	3
14	87	12	40	15	3
15	97	13	41	27	5
16	40	6	42	194	14
17	55	7	43	25	6
18	58	9	44	27	5
19	12	3	45	9	4
20	73	11	46	25	5
21	24	6	47	15	3
22	20	5	48	22	3
23	30	2	49	42	6
24	43	8	50	22	3
25	52	8	51	18	3
26	22	5	52	16	3

Table 4. Order Quantity and Reorder Point for Each Drug Item Under the Selected Aggregate Policy (Continued)

Item	Order Quantity	Reorder Point	Item	Order Quantity	Reorder Point
53	12	3	79	14	1
54	13	2	80	33	4
55	43	2	81	12	1
56	30	4	82	19	1
57	42	6	83	12	1
58	14	2	84	21	1
59	34	4	85	9	1
60	7	1	86	11	1
61	24	3	87	7	1
62	18	3	88	19	1
63	15	3	89	7	1
64	15	2	90	15	1
65	14	2	91	16	1
66	14	2	92	37	4
67	36	4	93	6	1
68	9	1	94	25	1
69	11	1	95	24	1
70	24	1	96	7	1
71	12	1	97	15	1
72	22	2	98	12	1
73	12	2	99	12	1
74	49	4	100	9	1
75	11	1	101	6	1
76	33	5	102	30	1
77	11	2	103	11	1
78	6	1	104	16	1

Table 4. Order Quantity and Reorder Point for Each Drug Item Under the Selected Aggregate Policy (Continued)

Item	Order Quantity	Reorder Point	Item	Order Quantity	Reorder Point
105	10	1	131	6	1
106	4	1	132	6	1
107	14	1	133	12	1
108	45	1	134	11	1
109	4	1	135	4	1
110	7	1	136	7	1
111	11	1	137	9	1
112	7	1	138	7	1
113	3	1			
114	9	1			
115	18	1			
116	14	1			
117	7	1			
118	28	2			
119	12	1			
120	3	2			
121	9	1			
122	4	1			
123	36	2			
124	6	1			
125	4	1			
126	3	1			
127	9	1			
128	4	1			
129	6	1			
130	34	1			

BIBLIOGRAPHY

1. Arrow, K. J., Harris, T., and Marschak, J., "Optimal Inventory Policy," Econometrica, XIX (1951), pp. 250-272.

2. Arrow, K. J., Karlin, S., and Scarf, H., Studies in the Mathematical Theory of Inventory and Production, Stanford University Press, Stanford, California, 1958.

3. Churchman, C. W., Ackoff, R. L., and Arnoff, E. L., Introduction to Operations Research, John Wiley and Sons, Inc., New York, 1957.

4. Davis, R. H., "Optimal Inventory Control Decision Rules for a Large Supply System," Operations Research, November-December, 1959, pp. 764-782.

5. Economic Almanac, Thomas Y. Crowell Company, New York, 1960.

6. Ekey, D. C., Talbird, J. B., and Newberry, T. L., "Inventory Reorder Points for Conditions of Variable Demand and Lead-Time," Journal of Industrial Engineering, XII (January-February, 1961), pp. 32-34.

7. Feeney, G., J., "Strategic Decisions in Inventory Control Operations," Management Science, II (1955), pp. 69-82.

8. Fetter, R. B. and Dalleck, W. C., Decision Models for Inventory Management, Richard D. Irwin, Inc., Homewood, Illinois, 1961.

9. Flagle, C. D., Huggins, W. H., and Roy, R. H., Operations Research and Systems Engineering, The Johns Hopkins Press, Baltimore, 1960.

10. Magee, J. F., Production Planning and Inventory Control, McGraw-Hill Book Company, Inc., New York, 1958.

11. Raymond, F. E., Quantity and Economy in Manufacture, McGraw-Hill Book Company, Inc., New York, 1931.

12. Sasieni, M., Yaspan, A., and Friedman, L., Operations Research, Methods and Problems, John Wiley and Sons, Inc., New York, 1959.

13. Vazsonyi, A., Scientific Programming in Business and Industry, John Wiley and Sons, Inc., New York, 1958.

14. Welch, W. E., Tested Scientific Inventory Control, Management Publishing Corporation, Greenwich, Connecticut, 1956.

15. Korn, G. A., Mathematical Handbook for Scientist and Engineers, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1961.

16. Miller, D. W., and Starr, M. K., Inventory Control: Theory and Practice, Prentice Hall, Englewood Cliff, N. J., 1962.

17. Buchan, J. and Koenigsberg, E., Scientific Inventory Management, Prentice Hall Book Company, Inc., Englewood Cliff, N. J., 1963.